

COMPACTIFICATIONS OF DISCRETE QUANTUM GROUPS

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ABSTRACT. Given a discrete quantum group (\mathcal{A}, δ) we construct Hopf $*$ -algebra \mathcal{AP} which is a unital $*$ -subalgebra of the multiplier algebra of \mathcal{A} . The structure maps for \mathcal{AP} are inherited from $M(\mathcal{A})$ and thus the construction yields a compactification of (\mathcal{A}, δ) which is analogous to the Bohr compactification of a locally compact group. This algebra has the expected universal property with respect to homomorphisms from multiplier Hopf algebras of compact type (and is therefore unique). This provides an easy proof of the fact that for a discrete quantum group with an infinite dimensional algebra the multiplier algebra is never a Hopf algebra.

1. INTRODUCTION

The research presented in this paper was motivated by investigations in the theory of C^* -algebraic quantum groups ([1, 3]). This theory is based on the theory of locally compact groups. Let G be a locally compact group and let $A = C_\infty(G)$ the algebra of all continuous functions vanishing at infinity on G . Then the algebra of bounded continuous functions on G is naturally isomorphic to the multiplier algebra $M(A)$ of A . A function $f \in M(A)$ is almost periodic on G if and only if $\delta(f) \in M(A) \otimes M(A)$, where δ is a morphism from A to $A \otimes A = C_\infty(G \times G)$ given by $(\delta f)(g_1, g_2) = f(g_1 g_2)$ and “ \otimes ” stands for the completed tensor product of C^* -algebras. The set of almost periodic functions is a commutative unital C^* -subalgebra of $M(A)$. This algebra is the algebra of continuous functions on the Bohr compactification of G (cf. [2, §41]). Our aim is to generalize this construction to discrete quantum groups in the framework of multiplier Hopf algebras introduced by Van Daele in [6].

We shall now describe the contents of the paper. In section 2 we gather necessary information about multiplier Hopf algebras and discrete quantum groups. Section 3 deals with the concept of slices with reduced functionals which is used in the next section. In Section 4 we define the algebra of almost periodic elements for a discrete quantum group and show that it is a Hopf $*$ -algebra. We also prove that this Hopf algebra has a universal property for morphisms of multiplier Hopf algebras of compact type into the original discrete quantum group. At the end of Section 4 we include some corollaries of our construction.

2. PRELIMINARIES

All algebras we shall consider in this paper will be over the field of complex numbers. Let \mathcal{A} be an algebra with non degenerate product. By $M(\mathcal{A})$ we shall denote the multiplier algebra of \mathcal{A} (see [6]). We shall use the fact that a multiplier m is determined by the linear map corresponding to multiplication by m from the left. Indeed: if $m, n \in M(\mathcal{A})$ and $ma = na$ for all $a \in \mathcal{A}$ then $(bm)a = b(ma) =$

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$b(na) = (bn)a$ for any b . By non degeneracy of the product in \mathcal{A} we have that right multiplication by n and m also agree as linear maps on \mathcal{A} . In other words, $m = n$ as multipliers.

Let \mathcal{A} and \mathcal{B} be algebras with non degenerate products. A homomorphism $\Phi: \mathcal{A} \rightarrow M(\mathcal{B})$ is non degenerate if $\Phi(\mathcal{A})\mathcal{B} = \mathcal{B}\Phi(\mathcal{A}) = \mathcal{B}$. Such a non degenerate homomorphism has a unique extension to a homomorphism of unital algebras $M(\mathcal{A}) \rightarrow M(\mathcal{B})$. In particular non degenerate homomorphisms can be composed just as usual homomorphisms between algebras. A composition of non degenerate homomorphisms is non degenerate.

We shall be concerned with the theory of multiplier Hopf algebras developed in [6, 8]. Recall that a multiplier Hopf algebra is a pair (\mathcal{A}, δ) consisting of an algebra \mathcal{A} with non degenerate product and a homomorphism $\delta: \mathcal{A} \rightarrow M(\mathcal{A} \otimes \mathcal{A})$ such that the maps

$$\begin{aligned} T_1 : \mathcal{A} \otimes \mathcal{A} \ni (a \otimes b) &\longmapsto \delta(a)(I \otimes b), \\ T_2 : \mathcal{A} \otimes \mathcal{A} \ni (a \otimes b) &\longmapsto (a \otimes I)\delta(b) \end{aligned}$$

have range equal to $\mathcal{A} \otimes \mathcal{A}$ and are bijective and such that the linear maps on $\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A}$ given by $T_2 \otimes \text{id}$ and $\text{id} \otimes T_1$ commute (in other words, δ is *coassociative*). It follows from this definition that δ is a non degenerate homomorphism.

In the fundamental reference [6] it is shown that for a multiplier Hopf algebra (\mathcal{A}, δ) there exists a multiplicative functional ε on \mathcal{A} called *counit* such that for all $a, b \in \mathcal{A}$

$$(\text{id} \otimes \varepsilon)((a \otimes I)\delta(b)) = ab = (\varepsilon \otimes \text{id})(\delta(a)(I \otimes b)).$$

This combined with bijectivity of either T_1 or T_2 implies that $\mathcal{A}^2 = \mathcal{A}$ which proves to be a very useful fact. Moreover $(\text{id} \otimes \delta)$ and $(\delta \otimes \text{id})$ are non degenerate homomorphisms $\mathcal{A} \otimes \mathcal{A} \rightarrow M(\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A})$. In particular we can consider the compositions $(\delta \otimes \text{id}) \circ \delta$ and $(\text{id} \otimes \delta) \circ \delta$ from \mathcal{A} to $M(\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A})$ and their extensions $M(\mathcal{A}) \rightarrow M(\mathcal{A} \otimes \mathcal{A} \otimes \mathcal{A})$. In this context the coassociativity means that the two latter maps are equal:

$$(\text{id} \otimes \delta)\delta = (\delta \otimes \text{id})\delta.$$

There also exists an antihomomorphism $\kappa: \mathcal{A} \rightarrow M(\mathcal{A})$ called *coinverse* (or *antipode*) such that for all $a, b, c \in \mathcal{A}$

$$\begin{aligned} m[(\text{id} \otimes \kappa)((a \otimes I)\delta(b))(I \otimes c)] &= a\varepsilon(b)c, \\ m[(a \otimes I)(\kappa \otimes \text{id})(\delta(b)(I \otimes c))] &= a\varepsilon(b)c. \end{aligned}$$

If (\mathcal{A}, δ) is a multiplier Hopf algebra and in addition \mathcal{A} is a $*$ -algebra and δ is a $*$ -homomorphism, the pair (\mathcal{A}, δ) is then called a multiplier Hopf $*$ -algebra. The counit is then a $*$ -character of \mathcal{A} and the coinverse satisfies $S(S(a)^*)^* = a$ for all $a \in \mathcal{A}$.

For a multiplier Hopf algebra (\mathcal{A}, δ) we can consider the pair (\mathcal{A}, δ') where δ' is a composition of δ with an extension to $M(\mathcal{A} \otimes \mathcal{A})$ of the flip map $a \otimes b \mapsto b \otimes a$. If (\mathcal{A}, δ') is also a multiplier Hopf algebra then (\mathcal{A}, δ) is called *regular*. The coinverse of a regular multiplier Hopf algebra is necessarily an antiisomorphism $A \rightarrow A$. In particular it is non degenerate (as a homomorphism from \mathcal{A} to the opposite algebra of $M(\mathcal{A})$) and extends to an antiisomorphism $M(\mathcal{A}) \rightarrow M(\mathcal{A})$.

A regular multiplier Hopf algebra (\mathcal{A}, δ) is said to be of *discrete type* if there exists a non zero element $h \in \mathcal{A}$ such that $ah = \varepsilon(a)h$ for all $a \in \mathcal{A}$ (cf. [9]). A typical example of a regular multiplier Hopf algebra of discrete type is a *discrete quantum group*, i.e. a multiplier Hopf $*$ -algebra (\mathcal{A}, δ) such that \mathcal{A} is a direct sum of full matrix algebras ([7, Def. 2.3]). It is well known ([6]) that multiplier Hopf $*$ -algebras are regular. Involutive structure will not play an important role in our considerations. Regularity will be much more important, but we shall keep the

$*$ -structure in order to be able to use results from the theory of Hopf $*$ -algebras ([5]).

Discrete quantum groups appeared first in [4, Sect. 3]. They were defined and studied in the framework of multiplier Hopf algebras in [7].

A functional φ on a multiplier Hopf algebra (\mathcal{A}, δ) is called *left invariant* if $(\text{id} \otimes \varphi)\delta(a) = \varphi(a)I$ for all $a \in \mathcal{A}$ ($(\text{id} \otimes \varphi)\delta$ is a map from \mathcal{A} to $M(\mathcal{A})$). Similarly a functional ψ on \mathcal{A} is *right invariant* if $(\psi \otimes \text{id})\delta(a) = \psi(a)I$ for all $a \in \mathcal{A}$. If φ is a left invariant functional for a regular multiplier Hopf algebra (\mathcal{A}, δ) then $\psi = \varphi \circ S$ is right invariant. The theory of regular multiplier Hopf algebras with invariant functionals is very rich. In particular the dual regular multiplier Hopf algebra can be defined as

$$\widehat{\mathcal{A}} = \{a\varphi : a \in \mathcal{A}\},$$

where φ is any non trivial left invariant functional. This definition does not depend on the choice of φ (which is in fact unique up to rescaling) and $\widehat{\mathcal{A}}$ can be endowed with a comultiplication $\widehat{\delta}$ dual to multiplication in \mathcal{A} and it becomes a regular multiplier Hopf algebra with invariant functionals. The biduality theorem [8, Thm. 4.12] says that the dual of $(\widehat{\mathcal{A}}, \widehat{\delta})$ is naturally isomorphic to (\mathcal{A}, δ) . The same assertions are true if we consider multiplier Hopf $*$ -algebras.

Let us conclude with a statement that all regular multiplier Hopf algebras of discrete type have non trivial invariant functionals ([9], see also [7]).

3. SLICES

Let \mathcal{A} be an algebra with non degenerate product. The space of all linear functionals on \mathcal{A} will be denoted by $\mathcal{A}^\#$. This vector space carries a natural \mathcal{A} -bimodule structure: for $f \in \mathcal{A}^\#$ and $a \in \mathcal{A}$

$$\begin{aligned} (af)(b) &= f(ba), \\ (fa)(b) &= f(ab) \end{aligned}$$

for all $b \in \mathcal{A}$. The space of all *reduced linear functionals* on \mathcal{A} is by definition

$$\mathcal{A}^* = \text{span} \{afb : f \in \mathcal{A}^\#, a, b \in \mathcal{A}\}.$$

Now assume that \mathcal{A} is a direct sum of matrix algebras. Any reduced functional on \mathcal{A} has a natural extension from \mathcal{A} to $M(\mathcal{A})$:

$$(afb)(m) = f(bma)$$

for any multiplier m . To see that this is well defined we must show that if $a_1, \dots, a_n, b_1, \dots, b_n \in \mathcal{A}$ and $f_1, \dots, f_n \in \mathcal{A}^\#$ then

$$\sum f_i(a_i c b_i) = 0$$

for all $c \in \mathcal{A}$ implies that

$$\sum f_i(a_i m b_i) = 0$$

for all $m \in M(\mathcal{A})$. Let e be the unit of the direct sum of the matrix algebras containing the elements a_1, \dots, a_n . Then $e \in \mathcal{A}$ and $a_i e = a_i$ for $1 \leq i \leq n$. Now we see that for any $m \in M(\mathcal{A})$

$$\sum f_i(a_i m b_i) = \sum f_i(a_i (em) b_i) = 0,$$

as $em \in \mathcal{A}$.

If \mathcal{B} is an algebra with non degenerate product then $\mathcal{B} \otimes \mathcal{A}$ is also an algebra with non degenerate product ([6, Lemma A.2]). For any $\xi \in \mathcal{A}^*$ and $\zeta \in \mathcal{B}^*$ the tensor product $\zeta \otimes \xi$ is a reduced functional on $\mathcal{B} \otimes \mathcal{A}$ and, as such, extends to $M(\mathcal{B} \otimes \mathcal{A})$.

Proposition 3.1. *Let \mathcal{A} and \mathcal{B} be algebras with non degenerate products and let Y be a multiplier of $\mathcal{B} \otimes \mathcal{A}$. Then for any $\xi \in \mathcal{A}^*$ there exists a unique multiplier $m \in M(\mathcal{B})$ such that*

$$(1) \quad (\zeta \otimes \xi)(Y) = \zeta(m)$$

for all $\zeta \in \mathcal{B}^*$. The multiplier m is called a right slice of Y with ξ and will be denoted by $(\text{id} \otimes \xi)(Y)$.

Proof. Let $\xi = afc$ with $a, c \in \mathcal{A}$ and $f \in \mathcal{A}^\#$. Define left and right multiplication by m as

$$\begin{aligned} m b_2 &= (\text{id} \otimes f)((I \otimes c)Y(b_2 \otimes a)), \\ b_1 m &= (\text{id} \otimes f)((b_1 \otimes c)Y(I \otimes a)). \end{aligned}$$

for $b_1, b_2 \in \mathcal{B}$. It remains to prove that

$$(b_1 m) b_2 = b_1 (m b_2)$$

and this follows from associativity of multiplication in $\mathcal{B} \otimes \mathcal{A}$:

$$\begin{aligned} b_1 [(\text{id} \otimes f)((I \otimes c)Y(b_2 \otimes a))] &= (\text{id} \otimes f)((b_1 \otimes I)(I \otimes c)Y(b_2 \otimes a)) \\ &= (\text{id} \otimes f)((b_1 \otimes c)Y(b_2 \otimes a)) \\ &= (\text{id} \otimes f)((b_1 \otimes c)Y(I \otimes a)(b_2 \otimes I)) \\ &= [(\text{id} \otimes f)((b_1 \otimes c)Y(I \otimes a))] b_2. \end{aligned}$$

Formula (1) holds, for if $\zeta = b_2 f' b_1$ with $b_1, b_2 \in \mathcal{B}$ and $f' \in \mathcal{B}^\#$ then

$$\begin{aligned} \zeta((\text{id} \otimes \xi)(Y)) &= f'(b_1(\text{id} \otimes \xi)(Y)b_2) \\ &= f'[(\text{id} \otimes f)((b_1 \otimes c)Y(b_2 \otimes a))] \\ &= (f' \otimes f)((b_1 \otimes c)Y(b_2 \otimes a)) \\ &= (b_2 f' b_1 \otimes a f c)(Y). \end{aligned}$$

Let n be another multiplier of \mathcal{B} such that for any $\zeta \in \mathcal{B}^*$

$$(\zeta)(n) = (\zeta \otimes \xi)(Y).$$

Then for any $g \in \mathcal{B}^\#$ and all $b, b' \in \mathcal{B}$ we have $g(bnb') = g(bmb')$ and it follows that $bnb' = bmb'$ for all $b, b' \in \mathcal{B}$. Therefore $b(nb' - mb') = 0$ for all $b \in \mathcal{B}$. Since $nb' - mb' \in \mathcal{B}$, this implies that $nb' = mb'$ for all b' and this means that $m = n$ as linear maps on \mathcal{B} . \square

4. COMPACTIFICATION OF A DISCRETE QUANTUM GROUP

The algebra of $n \times n$ matrices with complex entries will be denoted by M_n . Let \mathcal{A} be a direct sum of a family of such full matrix algebras. Then $M(\mathcal{A})$ is a product if the same family of matrix algebras. To see the isomorphism let $\mathcal{A} = \bigoplus_{\iota \in \mathbb{I}} \mathcal{A}_\iota$ with $\mathcal{A}_\iota = M_{n_\iota}$ for each $\iota \in \mathbb{I}$. The action of an infinite family $(m_\iota)_{\iota \in \mathbb{I}}$ with $m_\iota \in M_{n_\iota}$ on elements of \mathcal{A} is, of course, given by matrix multiplication in each summand. Conversely for a multiplier m of \mathcal{A} the corresponding family $(m_\iota)_{\iota \in \mathbb{I}}$ is obtained by setting $m_\iota = m e_\iota$ where e_ι is the unit of the matrix algebra \mathcal{A}_ι .

Lemma 4.1. *Let \mathcal{A} be a direct sum of matrix algebras and let x_1, \dots, x_N be linearly independent multipliers of \mathcal{A} . Then there exists an element $e \in \mathcal{A}$ such that $x_1 e, \dots, x_N e$ are linearly independent elements of \mathcal{A} . The element e may be chosen to be a central idempotent.*

Proof. To proceed we must introduce notation

$$\mathcal{A} = \bigoplus_{\iota \in \mathbb{I}} \mathcal{A}_\iota$$

with $\mathcal{A}_\iota = M_{n_\iota}$.

Suppose, contrary to the statement of the lemma, that for any $a \in \mathcal{A}$ the set $\{x_1 a, \dots, x_N a\}$ is not linearly independent. Let \mathcal{F} be the family of finite subsets the index set \mathbb{I} . Notice that the family \mathcal{F} is directed by inclusion. For any $F \in \mathcal{F}$ let e_F be the unit of the finite dimensional algebra $\bigoplus_{\iota \in F} \mathcal{A}_\iota$. We are under assumption that

$$(2) \quad \left(\begin{array}{l} \text{For any } F \in \mathcal{F} \text{ there exists a vector} \\ (\lambda_1^F, \dots, \lambda_N^F) \in \mathbb{C}^N \setminus \{0\} \\ \text{such that } \sum_{k=1}^N \lambda_k^F x_k e_F = 0. \end{array} \right)$$

For any $F \in \mathcal{F}$ we shall denote by V_F the (non zero) subspace of \mathbb{C}^N consisting of all vectors $(\lambda_1^F, \dots, \lambda_N^F)$ fulfilling the formula in (2).

Let $F_0, F \in \mathcal{F}$ with $F_0 \subset F$. Then for any $(\mu_1, \dots, \mu_N) \in V_F$ we have

$$\sum_{k=1}^N \mu_k x_k e_F = 0$$

and multiplying this relation from the right by e_{F_0} we obtain

$$\sum_{k=1}^N \mu_k x_k e_{F_0} = 0.$$

This means that $(\mu_1, \dots, \mu_N) \in V_{F_0}$ and consequently $V_F \subset V_{F_0}$. Let $V_\infty = \bigcap_{F \in \mathcal{F}} V_F$. A moment of reflection shows that this subspace is non zero.

Let $(\alpha_1, \dots, \alpha_N)$ be a non zero vector in V_∞ . then for any $F \in \mathcal{F}$ we have

$$\sum_{k=1}^N \alpha_k x_k e_F = 0.$$

Now for any $a \in \mathcal{A}$ there is an $F \in \mathcal{F}$ such that $a e_F = e_F a = a$. It follows that for any $a \in \mathcal{A}$

$$\sum_{k=1}^N \alpha_k x_k a = \sum_{k=1}^N \alpha_k x_k a e_F = \sum_{k=1}^N \alpha_k x_k e_F a = \left(\sum_{k=1}^N \alpha_k x_k e_F \right) a = 0.$$

In other words $\sum_{k=1}^N \alpha_k x_k = 0$, i.e. the elements $\{x_1, \dots, x_N\}$ are not linearly independent in $M(\mathcal{A})$.

This contradiction means that (2) is not true. Consequently there exists a central projection $e \in \mathcal{A}$ such that $x_1 e, \dots, x_N e$ are linearly independent. \square

Definition 4.2. Let (\mathcal{A}, δ) be a discrete quantum group. The set \mathcal{AP} of almost periodic elements for (\mathcal{A}, δ) is defined as

$$\mathcal{AP} = \{x \in M(\mathcal{A}) : \delta(x) \in M(\mathcal{A}) \otimes M(\mathcal{A})\}.$$

Theorem 4.3. Let (\mathcal{A}, δ) be a discrete quantum group and denote its coinverse by κ . Then \mathcal{AP} defined in Definition 4.2 is a unital $*$ -subalgebra of $M(\mathcal{A})$. Moreover we have

- 1.: $\delta(\mathcal{AP}) \subset \mathcal{AP} \otimes \mathcal{AP}$,
- 2.: $\kappa(\mathcal{AP}) = \mathcal{AP}$.

Proof. The set \mathcal{AP} is a unital $*$ -subalgebra of $M(\mathcal{A})$ as a pre image of a unital $*$ -subalgebra in a $*$ -homomorphism of unital algebras. Let us concentrate on the comultiplication. Fix $x \in \mathcal{AP}$. Then we can write

$$(3) \quad \delta(x) = \sum_{k=1}^N x_k \otimes y_k$$

with $(x_k)_{k=1,\dots,N}$ and $(y_k)_{k=1,\dots,N}$ in $M(\mathcal{A})$ and $(y_k)_{k=1,\dots,N}$ linearly independent. By coassociativity of δ we have

$$(4) \quad \sum_{k=1}^N \delta(x_k) \otimes y_k = \sum_{k=1}^N x_k \otimes \delta(y_k)$$

It follows that $\delta(x) \in M(\mathcal{A} \otimes \mathcal{A}) \otimes M(\mathcal{A}) \cap M(\mathcal{A}) \otimes M(\mathcal{A} \otimes \mathcal{A})$. For any reduced functional ξ on \mathcal{A} we can apply the map

$$(5) \quad \text{id}_{\mathcal{A} \otimes \mathcal{A}} \otimes \xi = \text{id}_{\mathcal{A}} \otimes (\text{id} \otimes \xi)$$

to $\delta(x)$. By (4) and (5) the value will lie in $M(\mathcal{A}) \otimes M(\mathcal{A})$.

Let e be a central idempotent in \mathcal{A} such that $y_1 e, \dots, y_N e$ are linearly independent (as in Lemma 4.1). Since the ideal generated by e is a finite dimensional direct summand of \mathcal{A} , it is easy to define functionals ξ_1, \dots, ξ_N on \mathcal{A} such that

$$\xi_l(y_k e) = \delta_{k,l}.$$

for $k, l = 1, \dots, N$ and

$$\xi_k = e \xi_k$$

for $k = 1, \dots, N$. Thus the functionals $(\xi_k)_{k=1,\dots,N}$ are reduced and taking slices of (4) with ξ_l we obtain

$$\begin{aligned} M(\mathcal{A}) \otimes M(\mathcal{A}) &\ni (\text{id}_{\mathcal{A} \otimes \mathcal{A}} \otimes \xi_l) \left(\sum_{k=1}^N \delta(x_k) \otimes y_k \right) \\ &= \sum_{k=1}^N \delta(x_k) \xi_l(y_k) = \delta(x_l). \end{aligned}$$

This means that for each $l \in \{1, \dots, N\}$ we have $x_l \in \mathcal{AP}$. In particular $\delta(x) \in \mathcal{AP} \otimes M(\mathcal{A})$. Now choosing maximal linearly independent subset out of the elements $\{x_1, \dots, x_N\}$ we can rearrange the sum (3) to have linearly independent elements of \mathcal{AP} making up the left leg of $\delta(x)$. Then we can show that all elements making up the right leg are contained in \mathcal{AP} using the same technique as we have done for the $\{x_1, \dots, x_N\}$. The final statement being

$$\delta(x) \in \mathcal{AP} \otimes \mathcal{AP},$$

which proves **1**.

In what follows we shall use techniques which have become standard in the theory of multiplier Hopf algebras. The relation between coinverse and comultiplication in a multiplier Hopf algebra is given in [6, Prop. 5.6]: for any $a, b \in \mathcal{A}$ we have

$$(6) \quad (I \otimes \kappa(b)) \delta(\kappa(a)) = (\kappa \otimes \kappa)(\delta'(a)(I \otimes b))$$

and δ' is the composition of δ with the flip automorphism (cf. Section 2).

Now our (\mathcal{A}, δ) is a regular multiplier Hopf algebra. In particular $\kappa \otimes \kappa$ is an antiisomorphism of $\mathcal{A} \otimes \mathcal{A}$ onto itself and it extends to an antiisomorphism of $M(\mathcal{A} \otimes \mathcal{A})$ onto itself. Thus the right hand side of (6) reads

$$(I \otimes \kappa(b))(\kappa \otimes \kappa)\delta'(a).$$

In view of non degeneracy of the product we conclude that

$$\delta(\kappa(a)) = (\kappa \otimes \kappa)\delta'(a)$$

for all $a \in \mathcal{A}$. Thus the two non degenerate antihomomorphisms $\delta \circ \kappa$ and $(\kappa \otimes \kappa) \circ \delta'$ agree on \mathcal{A} and their (unique) extensions agree on $M(\mathcal{A})$. With this identity point 2. is trivial. \square

In order to fully describe the rest of the structure of \mathcal{AP} we shall use the fact that the dual multiplier Hopf algebra of (\mathcal{A}, δ) is a Hopf $*$ -algebra. First let us observe that $M(\mathcal{A})$ can be identified with a subspace of $\hat{\mathcal{A}}^\#$. Indeed: any element of \mathcal{A} is of the form $a\varphi$ for some $a \in \mathcal{A}$ (with φ a left invariant functional on \mathcal{A}). Therefore any $m \in M(\mathcal{A})$ determines a linear map

$$\hat{\mathcal{A}} \ni a\varphi \mapsto \varphi(ma) \in \mathbb{C}.$$

Using the biduality theorem ([8, Thm. 4.12]) one can see that the structure of a $*$ -algebra on $M(\mathcal{A})$ coincides with the one obtained from $\hat{\mathcal{A}}$ on $\hat{\mathcal{A}}^\#$ as described in the beginning of [5, Sect. 3]. Now it is easy to see that \mathcal{AP} is a subspace of the space $\hat{\mathcal{A}}^0$ defined in [5, Def. 3.2]. Moreover using the biduality theorem one can see that the maps δ, ε and κ on \mathcal{AP} are exactly the restrictions of the corresponding maps making $\hat{\mathcal{A}}^0$ a Hopf $*$ -algebra (as in [5, Thm. 3.3]). In particular we have

$$\begin{aligned} m((\kappa \otimes \text{id})\delta(x)) &= \varepsilon(x)I, \\ m((\text{id} \otimes \kappa)\delta(x)) &= \varepsilon(x)I \end{aligned}$$

for all $x \in \mathcal{AP}$.

We can summarize the above considerations in the following theorem:

Theorem 4.4. *Let (\mathcal{A}, δ) be a discrete quantum group. Then the algebra \mathcal{AP} defined in Definition 4.2 with structure maps inherited from $M(\mathcal{A})$ is a Hopf $*$ -algebra.*

\mathcal{AP} is a unital $*$ -subalgebra of the multiplier algebra of \mathcal{A} . This means, in particular, that the inclusion $\mathcal{AP} \hookrightarrow M(\mathcal{A})$ is non degenerate. Therefore one should think that the quantum space underlying \mathcal{A} maps onto a dense subset of the compact quantum space underlying \mathcal{AP} . This map is a quantum group homomorphism. This situation is fully analogous to the classical construction [2, Sect. 41A–41C]. One has to keep in mind, however, that in contrast to the case of classical locally compact groups, our construction is purely algebraic and thus does not correspond exactly to the classical construction of Bohr compactification.

Another feature of the classical Bohr compactification \overline{G} of a locally compact group G is that for any compact group K and a continuous group homomorphism $\Psi: G \rightarrow K$ there exists a unique continuous homomorphism $\overline{\Psi}: \overline{G} \rightarrow K$ such that $\Psi = \overline{\Psi} \circ \alpha$ where α is the canonical homomorphism from G onto a dense subgroup of \overline{G} (cf. [10, Sect. 31]). The analogous statement is true in our framework. Let χ denote the inclusion of \mathcal{AP} into $M(\mathcal{A})$. Recall that a multiplier Hopf algebra of compact type is simply a Hopf algebra.

Theorem 4.5. *Let $(\mathcal{B}, \delta_{\mathcal{B}})$ be a multiplier Hopf algebra of compact type and let $\Phi: \mathcal{B} \rightarrow M(\mathcal{A})$ be a non degenerate homomorphism such that*

$$(7) \quad (\Phi \otimes \Phi) \circ \delta_{\mathcal{B}} = \delta \circ \Phi$$

(where we use the extension of δ to $M(\mathcal{A})$). Then there exists a unique Hopf algebra homomorphism $\overline{\Phi}: \mathcal{B} \rightarrow \mathcal{AP}$ such that $\Phi = \chi \circ \overline{\Phi}$.

Proof. Notice first that a non degenerate homomorphism from a unital algebra to $M(\mathcal{A})$ must be unital. It follows from (7) that the image of Φ is contained in \mathcal{AP} . Let us define $\overline{\Phi}$ as the same homomorphism as Φ , but considered now as a map from \mathcal{B} to \mathcal{AP} . This is clearly a Hopf algebra homomorphism and formula $\Phi = \chi \circ \overline{\Phi}$ is satisfied. The uniqueness of $\overline{\Phi}$ follows from the fact that χ is an embedding. \square

The standard reasoning shows that for a discrete quantum group (\mathcal{A}, δ) the Hopf algebra (\mathcal{AP}, δ) is the unique Hopf algebra with the universal property described in Theorem 4.5.

Let (\mathcal{A}, δ) be a discrete quantum group. In general it is not easy to find all elements of the algebra \mathcal{AP} , but some of them are easily described. Recall that an N dimensional corepresentation of (\mathcal{A}, δ) is an element $u \in M_N \otimes M(\mathcal{A})$ such that $(\text{id} \otimes \delta)u = u_{12}u_{13}$. If

$$u = \sum_{k,l=1}^N e_{k,l} \otimes u_{k,l},$$

where $(e_{k,l})_{k,l=1,\dots,N}$ are the matrix units in M_N , then

$$\delta(u_{k,l}) = \sum_{p=1}^N u_{k,p} \otimes u_{p,l} \in M(\mathcal{A}) \otimes M(\mathcal{A}).$$

This way we obtain

Proposition 4.6. *Let (\mathcal{A}, δ) be a discrete quantum group and let u be a finite dimensional corepresentation of (\mathcal{A}, δ) then the matrix elements of u belong to the algebra \mathcal{AP} .*

It is reasonable to conjecture that all almost periodic elements for a discrete quantum group are linear combinations of matrix elements of finite dimensional corepresentations. This is the case for classical groups (cf. [10, Sect. 31]).

It is possible, however, to point out elements that do not belong to \mathcal{AP} . We shall put this result in the following:

Proposition 4.7. *Let (\mathcal{A}, δ) be a discrete quantum group with \mathcal{A} infinite dimensional. Then the algebra \mathcal{AP} of almost periodic elements does not contain \mathcal{A} .*

Proof. Let h be the element of \mathcal{A} with the property that for any $a \in \mathcal{A}$

$$ah = ha = \varepsilon(a)h.$$

It is possible to show (cf. [8, Sect. 5]) that any element b of \mathcal{A} has a unique representation in the form

$$b = (\text{id} \otimes \omega)\delta(h),$$

where $\omega \in \widehat{\mathcal{A}}$ (notice that for discrete quantum groups $\widehat{\mathcal{A}} \subset \mathcal{A}^*$). However if $\delta(h)$ were in $M(\mathcal{A}) \otimes M(\mathcal{A})$ then the space of right slices of $\delta(h)$ with all functionals $\omega \in \widehat{\mathcal{A}}$ would have to be finite dimensional. \square

If (\mathcal{A}, δ) is a multiplier Hopf algebra then the comultiplication, coinverse and counit extend to the whole algebra $M(\mathcal{A})$. However in general $(M(\mathcal{A}), \delta)$ is not a Hopf algebra. More precisely we have the following corollary of Proposition 4.7.

Corollary 4.8. *Let (\mathcal{A}, δ) be a discrete quantum group with \mathcal{A} infinite dimensional. Then $(M(\mathcal{A}), \delta)$ is not a Hopf algebra.*

Proof. If $(M(\mathcal{A}), \delta)$ were a Hopf algebra, in other words, a multiplier Hopf algebra of compact type, the identity mapping $M(\mathcal{A}) \rightarrow M(\mathcal{A})$ would have to factor through the inclusion of \mathcal{AP} into $M(\mathcal{A})$. Proposition 4.7 tells us that this is not possible because \mathcal{AP} does not contain all elements of \mathcal{A} and consequently is not equal to $M(\mathcal{A})$. \square

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